Walk-powers and homomorphism bound of planar graphs

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Abstract

As an extension of the Four-Color Theorem it is conjectured that every planar graph of odd-girth at least $2k + 1$ admits a homomorphism to $PC_{2k} = (\mathbb{Z}_2^{2k}, \{e_1, e_2, \cdots, e_{2k}, J\})$ where e_i's are standard basis and J is all 1 vector. Noting that PC_{2k} itself is of odd-girth $2k + 1$, in this work we show that if the conjecture is true, then PC_{2k} is an optimal such a graph both with respect to number of vertices and number of edges. The result is obtained using the notion of walk-power of graphs and their clique numbers.

An analogous result is proved for bipartite signed planar graphs of unbalanced-girth $2k$. The work is presented on a uniform frame work of planar consistent signed graphs.

1 Introudction

1.1 Signed graphs, notation

Given a graph G, a *signature* on G is an assignment of signs, $+$ or $-$, to the edges. The set of negative edges is normally denoted by Σ and will normally be referred to as the signature. A *re-signing* is to change the signs of all edges incident to a given set of vertices or, equivalently, edges of an edge-cut. Two signatures are said to be equivalent if one can be obtained from the other by a re-signing. A graph G together with a set of signatures equivalent to Σ is called a signed graph and is denoted by $[G, \Sigma]$ where Σ is any member of the class of equivalent signatures. A signed cycle with an even (odd) number of negative edges is called *balanced* (*unbalanced*). It is easily observed that the balance of a cycle is invariant of re-signing. The *unbalanced-girth* of a signed graph is the shortest length of its unbalanced cycles. A *consistent* signed graph is a signed graph in which every balanced cycle is of even length and all unbalanced cycles are of a same parity. Thus there are two types of consistent signed graphs:

  i. when all unbalanced cycles are of odd length, it can be shown that this is the case if and only if $\Sigma \equiv E(G)$, such a signed graph will be called an *odd signed graph*;

  ii. when all unbalanced cycles are of even length, which will be the case if and only if G is bipartite, such a signed graph thus will be referred to as a *singed bipartite graph*.

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1.2 Homomorphisms and bounds

Given two signed graphs $[G, \Sigma]$ and $[H, \Sigma']$ we say there is a *homomorphism* of $[G, \Sigma]$ to $[H, \Sigma']$, denoted $[G, \Sigma] \to [H, \Sigma']$, if there is a signature Σ_1 of G equivalent to Σ and a mapping $\varphi : V(G) \to V(H)$ such that $xy \in E(G)$ implies $\varphi(x)\varphi(y) \in E(H)$ and $xy \in \Sigma_1$ if and only if $\varphi(x)\varphi(y) \in \Sigma'$. It is easy to prove that if $[G, \Sigma] \to [H, \Sigma']$, then unbalanced-girth of $[G, \Sigma]$ is at least as the unbalanced-girth of $[H, \Sigma']$. Give a class $\mathcal{C}$ of (signed) graphs we say a (signed) graph B *bounds* $\mathcal{C}$ if every member of $\mathcal{C}$ admits a homomorphism to B. For more on this subject we refer to [NRS13].

1.3 Signed Projective cubes

Projective cube of dimension d, denoted $\mathcal{PC}_d$, is the Cayley graph $(\mathbb{Z}_2^d, \{e_1, e_2, \cdots e_d, J\})$ where e_i's are the standard basis and J is the all 1 vector of relevant length (d here). It is obtained by identifying antipodal vertices of the hypercube of dimension $d+1$ or, equivalently, by adding edges between pairs of antipodal vertices of hypercube of dimension d. We define *singed projective cube* of dimension d, denoted $\mathcal{SPC}_d$, to be the singed graph obtained from $\mathcal{PC}_d$ by assigning $+$ to each edge corresponding to an e_i and $-$ to edges corresponding to J.

Projective cubes, also known as folded cubes, are well-studied graphs. We refer to [NRS13] and references there for some properties of signed projective cubes and for a proof of the following two theorems:

Theorem 1.1. *Signed projective cube of dimension d is a consistent signed graph and has unbalanced-girth $d + 1$.*

It follows that if a signed graph admits a homomorphism to a signed projective cube, it must be a consistent signed graph. Such a mapping then becomes equivalent to a packing problem as the following theorem claims:

Theorem 1.2. *A consistent signed graph (G, Σ) admits a homomorphism to $\mathcal{SPC}_d$ if and only if the edges set of G can be partitioned into $d + 1$ disjoint sets each of which induces a signature equivalent to Σ.*

The following conjecture, introduced in [N07] and [G05] (also see [NRS12]) is the focus of this work:

Conjecture 1.3. *Given $d \geq 2$, every planar consistent signed graph of unbalanced-girth $d + 1$ admits a homomorphism to $\mathcal{SPC}_d$.*

The conjecture is formed of two parts: for even values of d (by considering the signature in which all edges are negative) it claims that every planar graph of odd-girth at least $d+1$ admits a homomorphism to $\mathcal{PC}_d$. For odd values of d it says that every planar signed bipartite graph of unbalanced-girth at least $d+1$ admits a homomorphism to $\mathcal{SPC}_d$. Since $\mathcal{PC}_2$ is isomorphic to K_4, the very first case of this conjecture is the Four-Color Theorem.

1.4 Question of Nešetřil

This conjecture, for odd values of d was introduced in [N07] in relation to a question of J. Nešetřil who asked if there is a triangle-free graph to which every triangle-free planar graph admits a homomorphism. This question was answered in a larger frame work by P. Ossona de Mendez and J. Nešetřil which is stated based on the following notation.

Given a finite set $\mathcal{H}$ of connected graphs we use $Forb_h(\mathcal{H})$ to denote the class of all graphs which do not admit a homomorphism from any member of $\mathcal{H}$. Similarly, given a set $\mathcal{M}$ of graphs we use $Forb_m(\mathcal{M})$ to denote the class of all graphs that have no member of $\mathcal{M}$ as a minor.

Theorem 1.4. *[NO08] Given a finite set $\mathcal{M}$ of graphs and a finite set $\mathcal{H}$ of connected graphs, there is graph in $Forb_h(\mathcal{H})$ to which every graph in $Forb_h(\mathcal{H}) \cap Forb_m(\mathcal{M})$ admits a homomorphism.*

1.5   In this paper

The bound that are build using known proofs of this theorem are super exponential. To find the optimal bound in this theorem, in general, is a very difficult question. Indeed this question, in particular, contains the Hadwiger's conjecture simply by taking $\mathcal{M} = \mathcal{H} = \{K_n\}$. Conjecture 1.3 proposes a smaller bound for the case of $M = \{K_5, K_{3,3}\}$ and $H = \{C_{2k-1}\}$. For $k = 1$, (C_1 being a loop), since K_4 is a planar graph, it is the optimal answer by the Four-Color Theorem. For $k = 2$, it is proved in [N13] that $PC(4)$, known as the Clebsch graph, is the optimal bound. Here we prove that any bound of odd-girth $2k + 1$ for planar graphs of odd-girth $2k + 1$ has to have at least 2^{2k} vertcies each of degree at least $2k + 1$. This would imply that if Conjecture 1.3 holds, then $PC(2k)$ is an optimal bound. We prove an analogue result for the case of planar signed bipartite graphs, even though analogue of Theorem 1.4 for signed bipartite graphs is not proved yet.

2   Optimal bound for planar odd signed graphs

In this section we consider the first part of Conjecture 1.3. This case deals with odd signed graphs in which case one can assume all the edges are negative. Thus homomorphism problem here is simply a homomorphism of graphs.

To prove our result, in fact we prove a stronger claim in the following sense. Given a graph G and a positive integer k we define the k-th walk-power of G, denoted by $G^{(k)}$, to be a graph whose vertex set is also $V(G)$ with two vertcies x and y being adjacent if there is a walk of length k connecting x and y in G. This graph would be loopless only if k is odd and G has odd-girth at least $k + 2$, thus this will be the only case of interest for us in this work. If ϕ is a homomorphism of G to H, then it can easily be checked that ϕ is also a homomorphism of $G^{(k)}$ to $H^{(k)}$. Thus to prove our claim we will prove the following stronger result.

Theorem 2.1. *There is a planar graph G of odd-girth $2k + 1$ with $\omega(G^{(2k-1)}) \geq 2^{2k}$.*

To prove the theorem we will in fact construct an example of such a graph. This construction is based on the following local construction.

Lemma 2.2. *Let G be a graph obtained from subdividing edges of K_4 such that in a planar embedding of G each of the four faces is a cycle of length $2k + 1$. Then $G^{(2k-1)}$ is isomorphic to K_{4k}.*

Proof. Let a, b, c and d be the original vertices of the K_4 from which G is constructed. For $x, y \in \{a, b, c, d\}$ let P_{xy} be the subdivision of xy, and let t_{xy} be the length of this path. For an internal vertex w of P_{xy}, let P_{xw} (or P_{wx}) be the part of P_{xy} connecting w to x, let t_{xw} be the length of it.

We have

$$t_{ab} + t_{bc} + t_{ca} = t_{ab} + t_{bd} + t_{da}$$
$$= t_{ac} + t_{cd} + t_{da}$$
$$= t_{bc} + t_{cd} + t_{db}$$
$$= 2k + 1. \tag{1}$$

From equation 1 we have

$$t_{xy} = t_{wz} \text{ for } \{x, y, w, z\} = \{a, b, c, d\}, \tag{2}$$

that is to say the if all four faces are of a same length, then parallel edge of K_4 are subdivided the same number of times (the parity of the length of faces is not important for this claim, the even case will be used later).

Let u and v be a pair of vertices of G. If they are both vertices of a facial cycle of G, then there is a walk of length $2k - 1$ connecting them since each facial cycle is of length $2k + 1$. If there is no facial cycle of G containing both u and v, then they are internal vertices (after subdivision) of two distinct parallel edges of K_4, thus we may assume, without loss of generality, that u is a vertex of the path P_{ab} and v is a vertex of the path P_{cd}.

Note that by equation 2 we have

$$t_{au} + t_{bu} = t_{cv} + t_{dv}$$
$$= t_{ab} = t_{cd}. \tag{3}$$

If $t_{ab} = t_{cd}$ is even (odd respectively), then t_{au} and t_{bu} have the same parity (different parities respectively) and t_{cv} and t_{dv} have the same parity (different parities respectively). Moreover, since t_{cd} is even (odd respectively) and $t_{ac} + t_{cd} + t_{da} = 2k + 1$, t_{ac} and t_{ad} have different parities (same parity respectively).

Now one of the paths connecting u, v, say $P_{ua} \cup P_{ac} \cup P_{cv}$, is of length $t_{au} + t_{ac} + t_{cv}$, and another path, say $P_{ub} \cup P_{bd} \cup P_{dv}$, is of length $t_{bu} + t_{bd} + t_{dv}$. By (3) we have $(t_{bu} + t_{bd} + t_{dv}) + (t_{au} + t_{ac} + t_{cv}) = 2(t_{ab} + t_{bd})$, hence $t_{bu} + t_{bd} + t_{dv}$ and $t_{au} + t_{ac} + t_{cv}$ have a same parity. Furthermore, since $P_{ab} \cup P_{ad} \cup P_{bd}$ forms a facial cycle we have $t_{ab} + t_{ad} + t_{bd} = 2k + 1$, thus $2(t_{ab} + t_{bd}) = 4k + 2 - 2t_{bc} \leq 4k$.

Hence we have $min\{(t_{au} + t_{ac} + t_{cv}), (t_{bu} + t_{bd} + t_{dv})\} \leq 2k$. Similarly, we can show that $min\{(t_{au} + t_{ad} + t_{dv}), (t_{bu} + t_{bc} + t_{cv})\} \leq 2k$.

But note that $min\{(t_{au} + t_{ac} + t_{cv}), (t_{bu} + t_{bd} + t_{dv})\}$ and $min\{(t_{au} + t_{ad} + t_{dv}), (t_{bu} + t_{bc} + t_{cv})\}$ have different parities irrespective of the parity of $t_{ab} = t_{cd}$. Therefore, there is a walk of length $2k - 1$ from u to v. □

Proof of Theorem 2.1. Consider a K_4 on four vertices a, b, c and d. Let G_1 be a subdivision of this K_4 where edges ab and cd each are subdivided into $2k - 1$ edges. Thus G_1 is a subdivision of K_4 in which all the four faces are cycles of length $2k + 1$. Hence by Lemma 2.2 we have

$$\omega(G_1^{(2k-1)}) = |V(G_1)| = 4k.$$

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In the following we build a sequence of graphs $G_i, i = 1, 2, \cdots, 2k-1$, such that each G_{i+1}, $i \leq 2k-2$, contains G_i as a subgraph, G_{i+1} is planar and of odd-girth $2k+1$ and such that $\omega(G_{i+1}^{(2k-1)}) > \omega(G_i^{(2k-1)})$. At the final step we will have

$$\omega(G_{2k-1}^{(2k-1)}) \geq 2^{2k}.$$

We start with the following partial construction. Suppose G_i is built and let $P = uv_1v_2 \cdots v_r w$ be a maximal thread, that is, a path P connecting u and w such that all v_j's ($j \in \{1, 2, ..., r\}$) are of degree 2 in G_i but u and w are of degree at least 3. Furthermore, assume that P is either part of a path of length $2k-1$ connecting a and b or part of a path of length $2k-1$ connecting c and d.

Since P is a thread, if we add a new edge uw in G_i, the resulting graph will still be planar. So we add such an edge and subdivide it r times to obtain the new thread $P' = uv_1'v_2' \cdots v_r'w$. Consider a planar drawing of the graph in which P and P' form a facial cycle of length $2r+2$. In the face PP' connect v_1 and v_r' by a new edge. Subdivide this new edge $2k-r-1$ times (that is, into $2k-r$ edges, we draw it in dot line), so that each of the facial cycles containing the new thread is of length $2k+1$.
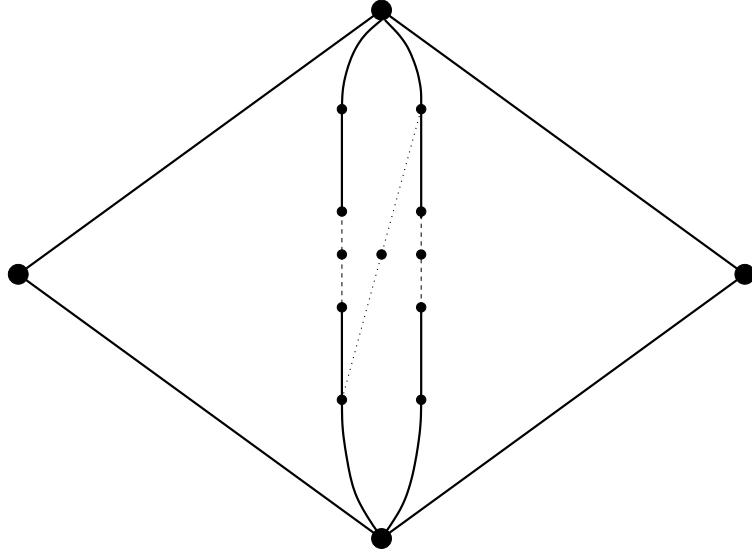

Figure 1: thread P

Denote by G_i' the resulting graph. We first note that G_i' is also of odd-girth $2k+1$. Now suppose a maximal clique W of $G_i^{(2k-1)}$ contains v_j of the thread P. Then we claim that $W \cup v_j'$ is also a clique of $G_i'^{(2k-1)}$.

To prove this let x be any vertex of W. If x is not in P, then consider a walk of length $2k-1$ from v_j to x. Each time this walk uses a part of P, replace it with the corresponding part from P' and this would give a walk of length $2k-1$ connecting x to v_j'.

If $x \in P$, then, without loss of generality, assume that P is part of a path of length $2k-1$ connecting a and b. Consider the subgraph induced by this path together with c, P' and the $v_1...v_r'$ thread we added to build G_i'. This induced subgraph is a subdivision of K_4 in which

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all the faces are cycles of length $2k + 1$. Thus, by Lemma 2.2 there is a walk of length $2k - 1$ connecting x and v'_j. Extending this argument we observe that if all vertices of P are in W, then $W \cup \{v'_1, v'_2, \cdots, v'_r\}$ is a clique in $G_i'^{(2k-1)}$.

Now we describe our general construction. At first we have G_1 on $4k$ vertices and two maximal threads. By Lemma 2.2 all the vertices of these two threads are parts of the unique clique of order $4k$ in $G_1^{(2k-1)}$. We apply the previously mentioned construction on both threads to build G_2 which will have four maximal threads each of length $2k - 1$ (we are only considering maximal threads that are part of a path of length $2k - 1$ connecting a, b or c, d, for example the $v'_{2k-2}v_1$-thread drawn in dot line of Figure 1 is not considered). There is a clique of order $4k + 2(2k - 2)$ in $G_2^{(2k-1)}$, and there are four maximal threads of length $2k - 2$, each is a part of a path of length $2k - 1$ either connecting a and b or c and d.

Continuing this construction, in general, there is a clique W_i of $G_i^{(2k-1)}$ $(2 \leq i \leq 2k - 1)$ which is of order $4k + \sum_{j=1}^{i-1} 2^j(2k - j - 1)$ and there are 2^i maximal threads of length $2k - i$ which each is a part of a path of length $2k - 1$ connecting a to b or c to d.

Note that G_i at each step is a planar graph of odd-girth $2k + 1$. The clique W_{2k-1} of $G_{2k-1}^{(2k-1)}$ has order equal to

$$
\begin{aligned}
4k + \sum_{j=1}^{2k-2} 2^j(2k - j - 1) &= 4k + (2k - 1)\sum_{j=1}^{2k-2} 2^j - 2\sum_{j=1}^{2k-2} j2^{j-1} \\
&= 4k + [(2k - 1)(2^{2k-1} - 2)] - \\
&\quad 2[(1 - 2^{2k-1}) - (-1)(2k - 1)2^{2k-2}] \\
&= 4k + (k2^{2k} - 4k - 2^{2k-1} + 2) - \\
&\quad (2 - 2^{2k} + k2^{2k} - 2^{2k-1}) \\
&= 2^{2k}.
\end{aligned}
$$

This completes the proof. $\square$

Corollary 2.3. *Let B be a graph of odd-girth $2k + 1$ to which every planar graph of odd-girth $2k + 1$ admits a homomorphism. Then $|V(B)| \geq 2^{2k}$. Furthermore, if B is minimal with this property, then $\delta(B) \geq 2k + 1$.*

Proof. Let G be a graph build in the previous theorem. Since G is of odd-girth $2k + 1$, by the assumption, it maps to B. Since B is also of odd-girth $2k + 1$, both $B^{(2k-1)}$ and $G^{(2k-1)}$ are simple graphs and $G^{(2k-1)} \to B^{(2k-1)}$. Hence $K_{2^{2k}} \subset B^{(2k-1)}$ which, in particular implies $|V(B)| \geq 2^{2k}$.

To prove the lower bound on minimum degree, we first introduce the following graph: let P be a path of length $2k$ on vertices $x_1, x_2, \cdots, x_{2k+1}$ connected in this order. Let P' be obtained from P by subdividing each edge $2k - 2$ times so that x_i is at distance $2k - 1$ from x_{i+1}. Let $y_1^i, y_2^i, \cdots y_{2k-2}^i$ be the new vertices subdividing $x_i x_{i+1}$ and connected in this order in P'. We add short cut edges so that the shortest odd walk between each x_i and x_j becomes of length $2k - 1$. These edges are $x_1 y_1^2, y_1^2 y_2^3, y_2^3 y_3^4, \cdots, y_{2k-2}^{2k} x_{2k+1}$. Now given a vertex u, the graph P_u is a graph which is formed from a disjoint copy of P' by connecting u to all x_i's. Note that the graph P_u is of odd-girth $2k + 1$ and that in $P_u^{(2k-1)}$ vertices of P (i.e., x_i's) induce a $(2k + 1)$-clique.

Now since B is minimal, there exists a planar graph G_B of odd-girth $2k + 1$ whose mappings to B are always onto. Let G_B^* be a new graph obtained from G_B by adding a P_u for each vertex

u of G_B. This new graph is also of odd-girth $2k + 1$, thus, by the choice of B, it maps to B. Let ϕ be such a mapping of G_B^* to B. This mapping induces a mapping of G_B to B. Thus each vertex v of B is image of a vertex u of G_B by the choice of G_B. But in the mapping G_B^* to B, all x_i's of P_u must map to distinct vertices all of whom are neighbours of $\phi(u) = v$.

$\square$

Note that since $PC(2k)$ is a $(2k + 1)$-regular graph on 2^{2k} vertices, it would be an optimal bound if Conjecture 1.3 holds.

3   Optimal bound for planar signed bipartite graphs

The development of the notion of homomorphisms for signed graphs has began very recently and, therefore, it is not yet known if an analogue of Theorem 1.4 would hold for the class of signed bipartite graphs. While we believe that would be the case, here we prove that SPC_k is the optimal bound for the signed bipartite case of Conjecture 1.3 if the conjecture holds.

To start, we introduce an analogue notion of walk-power. Let $[G, \Sigma]$ be a signed bipartite graph with (X, Y) being the partition of vertices. Given an even integer $r \geq 2$ we define $[G, \Sigma]^r$ to be a graph on $V(G)$ where a pair u, v of vertices are adjacent if the following conditions hold:

- u and v are in the same part of G;

- there are u, v-paths P_1 and P_2, each of length at most r, such that one has an odd number of negative edges and the other has even number of them.

Note that the second condition is independent of the choice of a representative signature. Furthermore $[G, \Sigma]^r$ is a graph (not signed) with no connection from X to Y.

We remark that these two conditions together are to say that: for any choice of an equivalent signature if u and v are identified then there would be an unbalanced cycle of even length at most r. That can be analogue to the definition of G^r for odd values of r where odd-girth of G is at least $r + 2$, in the following sense: first of all G can be regarded as a consistent signed graph $[G, E(G)]$; secondly for any choice of equivalent signature Σ of $[G, E(G)]^r$ if identifying pair u, v of vertices results in yet an odd (signed) graph (analogue of 1) but of unbalanced girth at most r (analogue of 2), then u and v are adjacent in $[G, E(G)]^r$. While $[G, E(G)]^r$ could be a proper subgraph of G^r, the claim and proof of Theorem 2.1 can be revised with this modified definition.

With the previous remark following lemma is easy to verify.

Lemma 3.1. *Let $[G, \Sigma]$ and $[H, \Pi]$ be two signed bipartite graphs and let ϕ be a homomorphism of $[G, \Sigma]$ to $[H, \Pi]$. Then for any positive even integer r, ϕ is also a homomorphism of $[G, \Sigma]^r$ to $[H, \Pi]^r$.*

Thus if both graphs are of unbalanced girth at least $r + 2$, then $[G, \Sigma]^r$ and $[H, \Pi]^r$ are both loopless, and, therefore, mapping ϕ would imply $\omega([G, \Sigma]^r) \leq \omega([H, \Pi]^r)$. Furthermore, assuming that G and H are both connected, since ϕ is also a mapping of G to H, it would preserve bipartition. Thus in what follows we will built a signed bipartite planar graph $[G, \Sigma]$ of unbalanced girth $2k$ such that each part of G contains a clique of size 2^{k-2} in $[G, \Sigma]^{2k-2}$.

To this end we start with the following lemma which is the signed bipartite analogue of Lemma 2.2.

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Lemma 3.2. *Let $[G, \Sigma]$ be a planar signed graph which is obtained from assigning a signature to a subdivision of K_4 in such a way that each of the four facial cycles is an unbalanced cycle of length $2k$. Then $[G, \Sigma]^{(2k-2)}$ is isomorphic to disjoint copies of $K_{(2k-1)}$ each induced on one part of the bipartite graph G.*

Proof. We consider a fixed signature Σ of $[G, \Sigma]$. We will use the same notations (P_{xy}, t_{xy} etc.) as in Lemma 2.2. Thus as proved in that lemma, parallel edges of K_4 are subdivided same number of times. Furthermore, repeating the same argument modulo 2, we can conclude that the number of negative edges in P_{xy} and the number of negative edges in P_{wz} have same parity for all $\{x, y, w, z\} = \{a, b, c, d\}$.

Let u and v be two vertices from same part of G (thus any path connecting u and v have even length). We would like to prove that they are adjacent in $[G, \Sigma]^{(2k-2)}$. If they belong to a same facial cycle, then the two paths connecting these two vertices in that (unbalanced) cycle satisfy the conditions and we are done. Hence, assume without loss of generality that $u \in P_{ab}$ and $v \in P_{cd}$.

Removing the edges of the parallel paths P_{ad} and P_{bc} will result in a cycle of length $4k - 2t_{ad}$ containing u, v. This implies:

$$(t_{ua} + t_{ac} + t_{cv}) + (t_{ub} + t_{bd} + t_{dv}) \leq 4k - 2$$
$$\Rightarrow \min\{(t_{ua} + t_{ac} + t_{cv}), (t_{ub} + t_{bd} + t_{dv})\} \leq 2k - 2. \tag{4}$$

Similarly by removing P_{ac} and P_{bd} we get

$$\min\{(t_{ua} + t_{ad} + t_{dv}), (t_{ub} + t_{bc} + t_{cv})\} \leq 2k - 2. \tag{5}$$

It remains to show that the two paths of equation (4) and (5) have different number of negative edges modulo 2. To see this note that union of any one of the two paths from (4) with a path from (5) covers a facial cycle exactly once and one a part of P_{ab} or P_{cd} twice. Since each facial cycle is unbalanced, our claim is proved. $\qquad \square$

We are now ready to present our general construction.

Theorem 3.3. *There exists a planar signed bipartite graph $[G, \Sigma]$ such that in $[G, \Sigma]^{(2k-1)}$ each part of G induces a clique of size at least 2^{2k-2}.*

Proof. Consider a K_4 on four vertices a, b, c and d. Let G_1 be a subdivision of this K_4 where edges ab and cd each are subdivided into $2k - 2$ edges. Note that G_1 is a connected bipartite graph and let V_1 and V_1' be its partite sets. Let Σ_1 be the signature with the new edge incident to a (created by the subdivision of ab) and the new edge incident to c (created by the subdivision of cd) being negative. Thus the signed bipartite graph $[G_1, \Sigma_1]$ is a subdivision of K_4 in which all the four faces are unbalanced cycles of length $2k$. Hence by Lemma 3.2 we know that each of V_1 and V_1' induces a clique of order $2k - 1$ in $[G_1, \Sigma_1]^{(2k-2)}$.

In the following we will build a sequence of signed graphs $[G_i, \Sigma_i]$, for $i \in \{1, 2, \cdots, 2k - 1\}$, such that each $[G_{i+1}, \Sigma_{i+1}]$, $i \leq 2k - 2$, contains $[G_i, \Sigma_i]$ as a subgraph. Moreover, the signed graph $[G_i, \Sigma_i]$ is a bipartite planar graph with unbalanced-girth $2k$ and partite sets V_i, V_i'. Let us denote the clique number of the graph induced by V_i (or V_i') from $[G_i, \Sigma_i]^{(2k-2)}$ by $f(i)$ (or $f'(i)$). Note that both the functions are strictly increasing and at the final step we will have

$$f(2k - 1), f'(2k - 1) \geq 2^{2k-2}.$$

We start with the following partial construction. Suppose $[G_i, \Sigma_i]$ is built and let $P = uv_1v_2 \cdots v_rw$ be a maximal thread. Furthermore, assume that P is either part of a path of length $2k - 2$ connecting a and b or part of a path of length $2k - 2$ connecting c and d.

Since P is a thread, if we add a new edge uw in $[G_i, \Sigma_i]$, the resulting graph will still be planar. So we add such an edge and subdivide it r times to obtain the new thread $P' = uv'_1v'_2 \cdots v'_rw$. Also we assign signs of the new edges in such a way that the edges uv'_1 and v_rw have a same sign, the edges v'_rw and uv_1 have a same sign and the edges $v'_iv'_{i+1}$ and $v_{r-i+1}v_{r-i}$ have a same sign.

Consider a planar drawing of the graph in which P and P' form a facial cycle of length $2r$. In the face PP' connect v_1 and v'_r by a new edge. Subdivide this new edge $2k - r - 2$ times (that is, into $2k - r - 1$ edges, we color them green), so that each of the facial cycles containing the new thread is of length $2k$. Choose signs of the edges of this new path in such a way that each of the facial cycles containing the new thread is unbalanced.

Let $[G'_i, \Sigma'_i]$ be the resulting signed graph. We first note that $[G'_i, \Sigma'_i]$ is also planar bipartite of unbalanced-girth $2k$. Now suppose that the vertices of P, indexed by odd (or even) numbers are all part of a maximal clique in the graph induced by one partite set of G_i in $[G_i, \Sigma_i]^{(2k-2)}$. Then the vertices of $P \cup P'$, indexed by odd (or even) numbers are all part of a maximal clique in the graph induced by the corresponding partite set of G'_i in $[G'_i, \Sigma'_i]^{(2k-2)}$. This can be proved by similar logic used in the proof of Theorem 2.1. The only difference is that to prove the above claim one needs to repeat the argument based on the parity of number of negative edges instead of the parity of number of edges and use Lemma 3.2 instead of Lemma 2.2.

Now we describe our general construction. At first we have $[G_1, \Sigma_1]$ on $4k - 2$ vertices and two maximal threads. By Lemma 3.2 the vertices, indexed by numbers with the same parity, of these two threads are parts of the unique clique of order $2k - 1$ in their respective components in $[G_1, \Sigma_1]^{(2k-2)}$. We apply the previously mentioned construction on both the threads to build $[G_2, \Sigma_2]$ which will have four maximal threads each of length $2k - 3$ (we are only considering maximal threads that are part of a path of length $2k - 2$ connecting a, b or c, d, the green threads are not considered). There are two disjoint cliques, each of order $(2k - 1) + (2k - 3)$ in $[G_2, \Sigma_2]^{(2k-2)}$, that is, $f(2) = f'(2) = (2k - 1) + (2k - 3)$, and there are four maximal threads of length $2k - 3$, each a part of a path of length $2k - 2$ either connecting a, b or connecting c, d.

Continuing this construction, in general, $f(i) = f'(i) = (2k - 1) + \sum_{j=1}^{i-1} 2^{j-1}(2k - j - 2)$ and there are 2^i maximal threads of length $2k - i - 1$ which are part of a path of length $2k - 2$ connecting a, b or connecting c, d.

Note that $[G_i, \Sigma_i]$ at each step is a planar bipartite signed graph of unbalanced-girth $2k$. Therefore

$$
\begin{aligned}
f(2k-1) = f'(2k-1) &= 2k - 1 + \sum_{j=1}^{2k-2} 2^{j-1}(2k - j - 2) = 2k - 1 + (k-1)\sum_{j=1}^{2k-2} 2^j - \sum_{j=1}^{2k-2} j2^{j-1} \\
&= 2k - 1 + [(k-1)(2^{2k-1} - 2)] - [(1 - 2^{2k-1}) - (-1)(2k-1)2^{2k-2}] \\
&= 2k - 1 + [k2^{2k-1} - 2k - 2^{2k-1} + 2] - [1 - 2^{2k-1} + k2^{2k-1} - 2^{2k-2}] \\
&= 2^{2k-2}.
\end{aligned}
$$

This completes the proof. □

Corollary 3.4. *If $[B, \Pi]$ is a minimal signed bipartite graph of unbalanced-girth $2k$ to which every planar signed bipartite graph of unbalanced-girth $2k$ admits a homomorphism, then $|V(B)| \geq 2^{2k-1}$ and $\delta(B) \geq 2k$.*

Proof. Let $[G, \Sigma]$ be the graph built in the previous theorem. Since $[G, \Sigma]$ is of unbalanced-girth $2k$, by the assumption, it maps to $[B, \Pi]$. Since $[B, \Pi]$ is also of unbalanced-girth $2k$, both $[B, \Pi]^{(2k-2)}$ and $[G, \Sigma]^{(2k-2)}$ are simple bipartite graphs and $[G, \Sigma]^{(2k-2)} \to [B, \Pi]^{(2k-2)}$. Hence there is a $K_{2^{2k-2}}$ in each part of $[B, \Pi]^{(2k-2)}$ which, in particular implies $|V(B)| \geq 2^{2k-1}$.

To prove the lower bound on minimum degree, note that since $[B, \Pi]$ is minimal, there exists a signed bipartite planar graph $[G_B, \Sigma_B]$ of unbalanced-girth $2k$ whose mappings to $[B, \Pi]$ are always onto. Now consider the graph $[G, \Sigma]$ built in the previous theorem. Note that in $[G, \Sigma]^{(2k-2)}$ all the $2k$ neighbours of a are adjacent to each other where a is one of the vertices of the K_4 that we started our construction with. Therefore, in any mapping of $[G, \Sigma]$ to $[B, \Pi]$ image of a must be of degree at least $2k$.

Now for each vertex x of $[G_B, \Sigma_B]$ add a vertex disjoint copy $[G_x, \Sigma]$ of $[G, \Sigma]$ and identify the vertex a of $[G_x, \Sigma]$ with x. Let $[G'_B, \Lambda_B]$ be the new graph. By the construction, $[G'_B, \Lambda_B]$ is also a signed bipartite planar graph of unbalanced-girth $2k$. Hence it maps to $[B, \Pi]$. In any such mapping, by the choice of $[G_B, \Sigma_B]$ and construction of $[G'_B, \Lambda_B]$, each vertex of $[B, \Pi]$ is an image of a in a mapping of $[G, \Sigma]$ to $[B, \Pi]$. Thus each vertex of $[B, \Pi]$ has degree at least $2k$. $\qquad\square$

4   Concluding remarks

P. Seymoure has conjectured in [S75] that the edge-chromatic number of a planar multi-graph is equal to it fraction edge-chromatic number. It turns out that the restriction of this conjecture for k-regular multigraph can be proved if and only if Conjecture 1.3 is proved for this value of $k = d - 1$. This special case of Seymour conjecture is proved for $k \leq 8$ in series of work using induction and the Four-Color Theorem in [G12] ($k = 4, 5$), [DKK] ($k = 6$), [E11] ($k = 7$) and [CES12] ($k = 8$). Thus Conjecture 1.3 is verified for $d \leq 7$. Hence we have the following corollary.

Theorem 4.1. *For $d \leq 7$ the signed graph SPC_d is the smallest consistent graph (both in terms of number of vertices and edges) of unbalanced-girth $d + 1$ which bounds all consistent planar signed graphs of unbalanced-girth at least $d + 1$.*

B. Guenin has proposed a strengthening of Conjecture 1.3 by replacing the condition of planarity with no $(K_5, E(K_5))$-minor.

For further generalization one can consider the following general question:

Problem 4.2. *Given d and r, $d \geq r$ and $d = r \pmod 2$ what is the optimal bound of unbalanced girth r which bounds all consistent signed graph of unbalanced-girth d with no $(K_n, E(K_n))$-minor?*

We do not yet know of existence of such a bound in general. For $n = 3$, consistent signed graphs with no $(K_n, E(K_n))$-minor are bipartite graphs with all edges positive, and, therefore, bounded by K_2. For $n = 5$ if the input and target graph are both of unbalanced-girth $d + 1$, then our work and Geunin's extension of Conjecture 1.3 proposes projective cubes as the optimal solutions. For $d = r = 3$, the answer would be K_{n-1} if Odd Hadwiger conjecture is true. For the case of $n = 4$ some partial answers are given by F. Foucaud and first author. For all other cases there is not even a conjecture yet.

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